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Spider-web solutions of the coupled KP equation

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Abstract

The coupled Kadomtsev–Petviashvili (cKP) equation possesses N-soliton solutions with more parametric freedom than the solitons of the usual KP equation. Its solutions can therefore be expected to model far more complex interactions than their KP counterparts. The existence of 'web'-like structures (on a finite scale) for cKP solutions (Isojima S, Willox R and Satsuma J 2002 *J. Phys. A: Math. Gen.* **35** 6893–6909) is a manifestation of this greater freedom. In this paper, we propose a new method to analyse the behaviour of solitons which we demonstrate in some examples. In addition we discuss 'essentially three-body collisions', described by a *two*-soliton solution of the cKP equation.

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1. Introduction

The coupled Kadomtsev–Petviashvili (cKP) equation

$$\left(D_x^4 - 4D_xD_t + 3D_y^2\right)\tau\cdot\tau = 24\hat{\sigma}\sigma \tag{1}$$

$$\left(D_x^3 + 2D_t - 3D_xD_y\right)\hat{\sigma}\cdot\tau = 0 \tag{2}$$

$$\left(D_x^3 + 2D_t + 3D_xD_y\right)\sigma\cdot\tau = 0 \tag{3}$$

was first proposed as the soliton system obtained by coupling the KP and the Davey–Stewartson equation (although in a slightly different scaling of the independent variables) [1]. The above equations are presented in terms of the Hirota bilinear operators D_x, D_y and D_t, defined as

$$D_t^kD_y^mD_x^n f\cdot g := \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k \times \left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m \times \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n \times f(x,y,t)g(x',y',t')\Big|_{\substack{x'=x\\ y'=y\\ t'=t}}. \tag{4}$$

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Equations (1)–(3) can be transformed into the following coupled nonlinear partial differential equations:

$$(4u_t - 6uu_x - u_{xxx})_x - 3u_{yy} + 24(v\hat{v})_{xx} = 0 \tag{5}$$

$$2\hat{v}_t + 3u\hat{v}_x + \hat{v}_{xxx} - 3\left(\hat{v}_{xy} + \hat{v}\int^x u_y\,\mathrm{d}x\right) = 0 \tag{6}$$

$$2v_t + 3uv_x + v_{xxx} + 3\left(v_{xy} + v\int^x u_y\,\mathrm{d}x\right) = 0 \tag{7}$$

by means of the dependent variable transformation

$$u = 2(\log\tau)_{xx} \tag{8}$$

$$\hat{v} = \hat{\sigma}/\tau \tag{9}$$

$$v = \sigma/\tau. \tag{10}$$

Remark that when $\hat{\sigma}\sigma = 0$ (and consequently $\hat{v}v = 0$) we obtain the standard KP equation and its bilinear form from the above equations.

It is well known that the cKP equation has *N*-soliton solutions which can be expressed in terms of Pfaffians [2] and that, as such, it is actually part of a hierarchy of integrable equations [2–4] possessing Pfaffian-type solutions. Recently it turned out that the cKP equation (5)–(7) (exactly as it stands) exhibits a profound connection with matrix integrals over the orthogonal and symplectic (Hermitian) ensembles (see e.g. [5, 6]) and its solutions might therefore find applications in the context of random matrix theory [7]. Although the cKP equation, in this sense, definitely merits a place among the classic soliton equations, the detailed behaviour of its soliton solutions was only clarified quite recently [8]. In [8] we discussed—by means of graphic simulations—the existence of 'web'-like structures for certain soliton solutions of the cKP equation. These solutions were named 'spider-web solutions'. Here we shall study the behaviour of the spider-web solutions of the cKP equation in considerable detail, using analytical tools instead of relying on graphical simulation. In section 2, we list the functional forms of the soliton solutions of the cKP equation. We shall introduce a new method in section 3 for the analysis of soliton solutions and then apply this method to some specific examples of spider-web solutions. In section 4, we will study the asymptotic behaviour of the two-soliton solution of the cKP equation and show that the two-soliton solution of the cKP equation either behaves as a spider-web solution or describes what we will call essentially three-body collisions.

2. The soliton solutions of the cKP equation

In the following, let p_j and q_j ($j = 1, 2, \ldots, 2N$) be arbitrary parameters characterizing the behaviour of the solitons. Moreover, let α_j, β_j ($j = 1, 3, \ldots, 2N-1$) be (arbitrary) phase constants governing the relative positions of the solitons. We can then define the phase functions ξ_p and $\theta_{p_j p_k}$

$$\xi_p := px + p^2 y + p^3 t \tag{11}$$

$$\theta_{p_j p_k} := \xi_{p_j} + \xi_{p_k} \tag{12}$$

and the interaction factors $\mathrm{A}(p_j, p_{j+1}; q_k, q_{k+1})$ ($j, k = 1, 3, \ldots, 2N-1$)

$$\mathrm{A}(p_j, p_{j+1}; q_k, q_{k+1}) := \frac{(p_j - p_{j+1})(q_k - q_{k+1})}{(p_j - q_k)(p_j - q_{k+1})(p_{j+1} - q_k)(p_{j+1} - q_{k+1})}. \tag{13}$$

For example, the one-soliton solution of the cKP equation can be represented by the following three functions:

$$\tau = 1 + \alpha_1\beta_1 \mathrm{A}(p_1, p_2; q_1, q_2) \exp\left(\theta_{p_1p_2} - \theta_{q_1q_2}\right) \tag{14}$$

$$\hat{\sigma} = \alpha_1(p_1 - p_2) \exp\left(\theta_{p_1p_2}\right) \tag{15}$$

$$\sigma = \beta_1(q_1 - q_2) \exp\left(-\theta_{q_1q_2}\right). \tag{16}$$

From (14)–(16), one can obtain the functional form of u, v, $\hat{v}$ corresponding to the one-soliton solution of the cKP equation:

$$\begin{aligned} u &= 2(p_1 + p_2 - q_1 - q_2)^2 \frac{\exp\left(\theta_{p_1p_2} - \theta_{q_1q_2} + \delta\right)}{\left(1 + \exp\left(\theta_{p_1p_2} - \theta_{q_1q_2} + \delta\right)\right)^2} \\ &= \frac{(p_1 + p_2 - q_1 - q_2)^2}{2} \operatorname{sech}^2\left[\frac{1}{2}\left(\theta_{p_1p_2} - \theta_{q_1q_2} + \delta\right)\right] \end{aligned} \tag{17}$$

$$\hat{v} = \frac{\alpha_1(p_1 - p_2)\exp(\theta_1)}{1 + \exp(\theta_1 - \theta_2 + \delta)} = \alpha_1(p_1 - p_2)\frac{1}{\exp(-\theta_1) + \exp(-\theta_2 + \delta)} \tag{18}$$

$$v = \frac{\beta_1(q_1 - q_2)\exp(-\theta_2)}{1 + \exp(\theta_1 - \theta_2 + \delta)} = \beta_1(q_1 - q_2)\frac{1}{\exp(\theta_2) + \exp(\theta_1 + \delta)} \tag{19}$$

where $\delta := \log \alpha_1\beta_1 \mathrm{A}(p_1, p_2; q_1, q_2)$. Note that the functional form of u is that of a typical (bell-shaped) KP sech^2-soliton. On the other hand, v and $\hat{v}$ will always diverge (rapidly) in some direction. Their product $v\hat{v}$, however, has the same functional form as the variable u, up to a constant multiple:

$$\begin{aligned} v\hat{v} &= \frac{\alpha_1\beta_1(p_1 - p_2)(q_1 - q_2)\exp\left(\theta_{p_1p_2} - \theta_{q_1q_2}\right)}{\left(1 + \exp\left(\theta_{p_1p_2} - \theta_{q_1q_2} + \delta\right)\right)^2} \\ &= \frac{(p_1 - q_1)(p_1 - q_2)(p_2 - q_1)(p_2 - q_2)}{4} \operatorname{sech}^2\left[\frac{1}{2}\left(\theta_{p_1p_2} - \theta_{q_1q_2} + \delta\right)\right]. \end{aligned} \tag{20}$$

Typical behaviour of the dependent variables $u, v, \hat{v}$ and $v\hat{v}$ is shown in figure 1. In the case of the N-soliton solution, it will still be true that we find similar rational expressions of exponential functions for u and $v\hat{v}$, but for the coefficients (of the exponential functions) in their numerators (in the case of the one-soliton solution this obviously results in $v\hat{v}$ being equal to u up to a constant multiple). What this really means of course is that both u and $v\hat{v}$ describe similar interactions of solitons, except for their amplitudes or relative phases. However, for computational reasons we shall limit our discussions to the properties of the u-field as it only depends on a single function τ, instead of three different functions for $v\hat{v}$.

The functional forms of $\tau, \sigma, \hat{\sigma}$, corresponding to the two-soliton solution of the cKP equation, are

$$\begin{aligned} \tau = 1 &+ \alpha_1\beta_1 \mathrm{A}(p_1, p_2; q_1, q_2)\exp\left(\theta_{p_1p_2} - \theta_{q_1q_2}\right) + \alpha_3\beta_3 \mathrm{A}(p_3, p_4; q_3, q_4)\exp\left(\theta_{p_3p_4} - \theta_{q_3q_4}\right) \\ &+ \alpha_1\beta_3 \mathrm{A}(p_1, p_2; q_3, q_4)\exp\left(\theta_{p_1p_2} - \theta_{q_3q_4}\right) \\ &+ \alpha_3\beta_1 \mathrm{A}(p_3, p_4; q_1, q_2)\exp\left(\theta_{p_3p_4} - \theta_{q_1q_2}\right) \\ &+ \alpha_1\beta_1\alpha_3\beta_3 \frac{\prod_{1\leqslant i<j\leqslant 4}\{(p_i - p_j)(q_i - q_j)\}}{\prod_{i,j=1}^{4}(p_i - q_j)} \exp\left(\theta_{p_1p_2} + \theta_{p_3p_4} - \theta_{q_1q_2} - \theta_{q_3q_4}\right) \end{aligned} \tag{21}$$

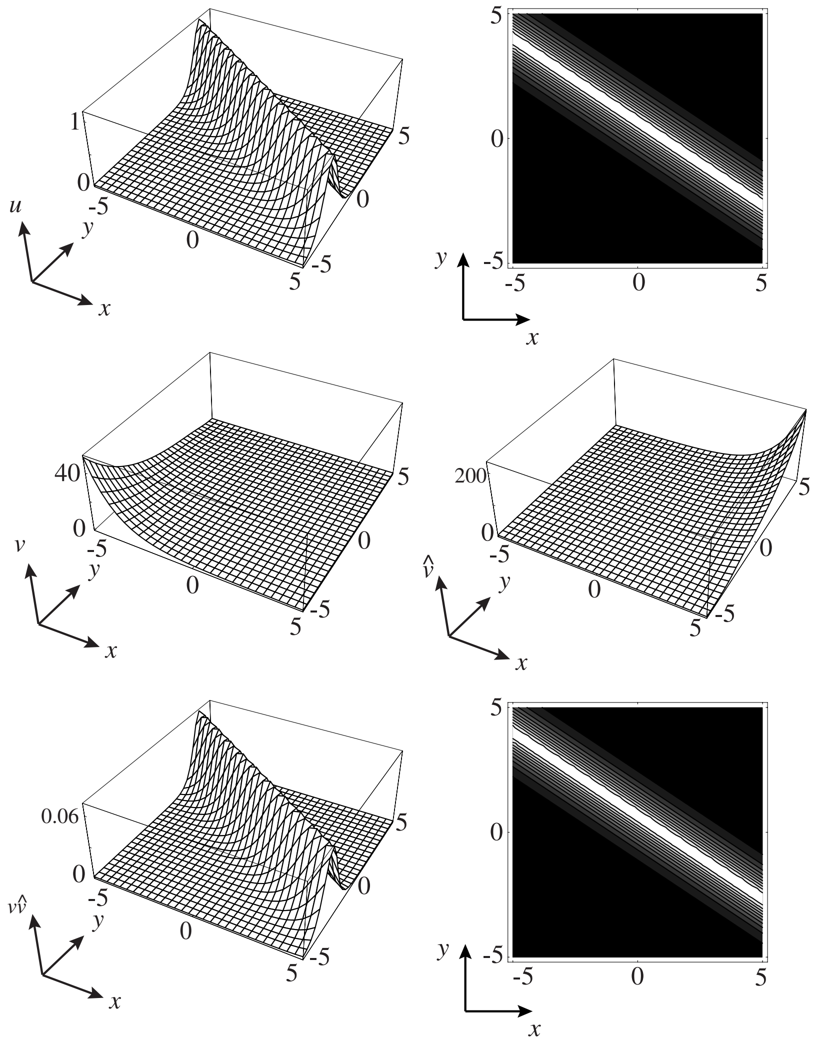

Figure 1. An example of the one-soliton solution of the cKP equation (given as a 3D plot and a contour plot).

$$\begin{aligned}
\hat{\sigma} = {} & \alpha_1(p_1 - p_2)\exp\left(\theta_{p_1p_2}\right) + \alpha_3(p_3 - p_4)\exp\left(\theta_{p_3p_4}\right) \\
& + \alpha_1\alpha_3\beta_1 \frac{(q_1 - q_2)\prod_{1\leqslant i<j\leqslant 4}(p_i - p_j)}{\prod_{1\leqslant i\leqslant 4,\, j=1,2}(p_i - q_j)} \exp\left(\theta_{p_1p_2} + \theta_{p_3p_4} - \theta_{q_1q_2}\right) \\
& + \alpha_1\alpha_3\beta_3 \frac{(q_3 - q_4)\prod_{1\leqslant i<j\leqslant 4}(p_i - p_j)}{\prod_{1\leqslant i\leqslant 4,\, j=3,4}(p_i - q_j)} \exp\left(\theta_{p_1p_2} + \theta_{p_3p_4} - \theta_{q_3q_4}\right)
\end{aligned} \tag{22}$$

$$\begin{aligned}
\sigma = {} & \beta_1(q_1 - q_2)\exp\left(-\theta_{q_1q_2}\right) + \beta_3(q_3 - q_4)\exp\left(-\theta_{q_3q_4}\right) \\
& + \alpha_1\beta_1\beta_3 \frac{(p_1 - p_2)\prod_{1\leqslant i<j\leqslant 4}(q_i - q_j)}{\prod_{i=1,2,\, 1\leqslant j\leqslant 4}(p_i - q_j)} \exp\left(\theta_{p_1p_2} - \theta_{q_1q_2} - \theta_{q_3q_4}\right) \\
& + \alpha_3\beta_1\beta_3 \frac{(p_3 - p_4)\prod_{1\leqslant i<j\leqslant 4}(q_i - q_j)}{\prod_{i=3,4,\, 1\leqslant j\leqslant 4}(p_i - q_j)} \exp\left(\theta_{p_3p_4} - \theta_{q_1q_2} - \theta_{q_3q_4}\right).
\end{aligned} \tag{23}$$

For later convenience, we shall define the following simple notation ζ_{jk} for the phase functions in the two-soliton τ:

$$\zeta_{jk} := \theta_{p_j p_{j+1}} - \theta_{q_k q_{k+1}} + \log\{\alpha_j \beta_k \mathrm{A}(p_j, p_{j+1}; q_k, q_{k+1})\}. \tag{24}$$

Furthermore, we denote the last term in τ of the two-soliton solution as $\mathrm{e}^{\zeta_{1133}}$. In this notation, the functional form of the dependent variable τ of the two-soliton solution takes the simple form

$$\tau = 1 + \mathrm{e}^{\zeta_{11}} + \mathrm{e}^{\zeta_{33}} + \mathrm{e}^{\zeta_{13}} + \mathrm{e}^{\zeta_{31}} + \mathrm{e}^{\zeta_{1133}}. \tag{25}$$

It should be pointed out that multiplication of $\tau, \sigma, \hat{\sigma}$ by a factor $\exp\varphi$, for an arbitrary linear function $\varphi(x, y, t)$, is one of the (fundamental) gauge transformations for the bilinear equations (1)–(3). Hence, we can view all functions obtainable through such a transformation as gauge equivalent. Moreover, due to the particular expressions (8)–(10), u, v and $\hat{v}$ are invariant under such transformations and, in what follows, we shall therefore treat different gauges of the function τ as one and the same. Making use of this gauge equivalence we see that, for example in the case of the two-soliton solution, there is a maximum of 15 different phases that can possibly appear in τ (or in one of its gauge equivalent expressions):

$$\begin{array}{lll}
\begin{cases}\zeta_{11} \\ \zeta_{1133} - \zeta_{33}\end{cases} & \begin{cases}\zeta_{33} \\ \zeta_{1133} - \zeta_{11}\end{cases} & \begin{cases}\zeta_{13} \\ \zeta_{1133} - \zeta_{31}\end{cases} \\
\begin{cases}\zeta_{31} \\ \zeta_{1133} - \zeta_{13}\end{cases} & \begin{cases}\zeta_{11} - \zeta_{13} \\ \zeta_{31} - \zeta_{33}\end{cases} & \begin{cases}\zeta_{11} - \zeta_{31} \\ \zeta_{13} - \zeta_{33}\end{cases} \\
\ \zeta_{11} - \zeta_{33} & \ \zeta_{13} - \zeta_{31} & \ \zeta_{1133}
\end{array} \tag{26}$$

(phases grouped together in a bracket only differ by a constant). Obviously, all of these can be written as a difference $\zeta_1 - \zeta_2$ of two of the six fundamental phases that appear in (25): $\zeta_{1133}, \zeta_{31}, \zeta_{13}, \zeta_{33}, \zeta_{11}$ or 0 (i.e. the ‘phase’ of the constant term).

3. A new method for analysing the behaviour of solitons

3.1. Description of the method

In our previous paper [8], we presented ‘spider-web solutions’ to the cKP equation. The interesting thing about these solutions is that, e.g., in the case of the two-soliton, the interaction waves form a tetragon on a finite domain, with an area that changes over time (see figure 2). Moreover, more complex patterns of polygons will appear as the number of solitons increases (see figure 3). Unfortunately however, up to this day almost no methods for the analysis of a multi-soliton solution on a finite domain are known (with the possible exception of [9]): the asymptotic analysis ordinarily used to investigate the behaviour of solitons being only applicable on an infinite domain. The method we shall present here, however, enables us to clarify the behaviour of solitons in the entire xy-plane (or any, possibly finite, sub-domain thereof) at arbitrary fixed times. In general it will be shown that all waves that appear in a typical soliton interaction pattern can be approximated by one-solitons, regardless of the nature of the interactions (in contrast to such analysis as found in [10]).

Let us consider the case of the two-soliton solution as an example. For simplicity, we represent the phase functions at fixed t as

$$\zeta_{jk} = P_{jk}x + Q_{jk}y + \delta_{jk} \tag{27}$$

$$P_{jk} := p_j + p_{j+1} - q_k - q_{k+1} \tag{28}$$

$$Q_{jk} := {p_j}^2 + {p_{j+1}}^2 - {q_k}^2 - {q_{k+1}}^2 \qquad (j, k = 1, 3) \tag{29}$$

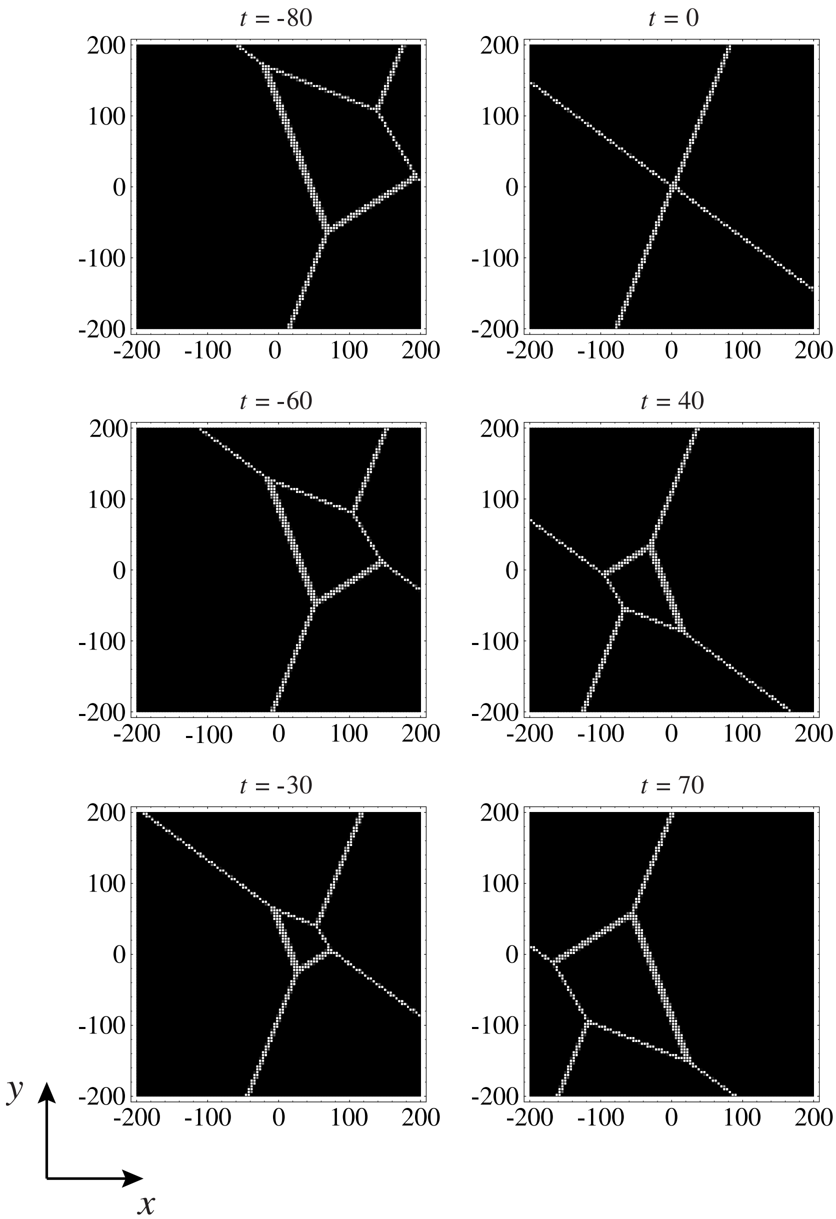

Figure 2. An example of the spider-web solution (the two-soliton solution as a density plot).

$$\zeta_{1133} = P_{1133}x + Q_{1133}y + \delta_{1133} \tag{30}$$

$$P_{1133} := P_{11} + P_{33} = P_{13} + P_{31} \tag{31}$$

$$Q_{1133} := Q_{11} + Q_{33} = Q_{13} + Q_{31}. \tag{32}$$

Note that the δ_{jk} and δ_{1133} are constants that depend on some previously chosen values of the parameters and some fixed t.

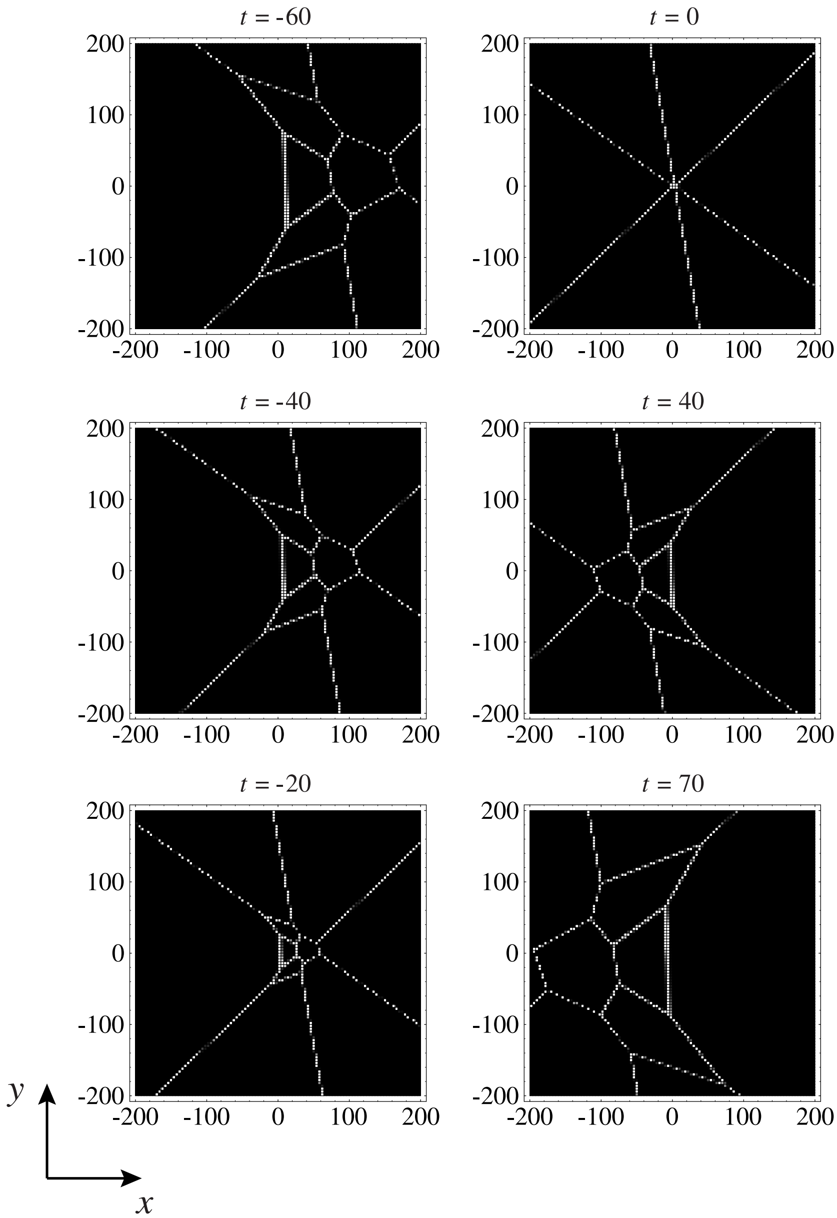

Figure 3. An example of a spider-web solution obtained from a three-soliton, given as a density plot.

The method we propose for the analysis of such (and more general) solutions boils down to a fine-tuning of the usual asymptotic analysis one carries out to study soliton solutions. It can be summarized as follows:

Step 0. Fix the independent variable t (arbitrarily).

Step 1. Choose one phase (denoted as ζ) from all possible phases that can appear in the solution and assume that this particular phase is small (almost zero). For example, we choose

$\zeta \equiv \zeta_{11}$ among the phases of (26) and we consider a region in the xy-plane near the line $\zeta_{11} = 0$. For convenience, we shall define $\eta_{11} := \zeta_{11}/P_{11}$ (note that the coefficient of x in η_{11} is 1).

Step 2. Express the x-dependence of all other phases in terms of the phase chosen in step 1 and calculate the 'remainders'. More precisely, starting with the phase ζ_{33}, express it as

$$\zeta_{33} = P_{33}\eta_{11} + (Q_{33} - P_{33}Q_{11}/P_{11})y + (\delta_{33} - P_{33}\delta_{11}/P_{11}).$$

We call the contribution $(Q_{33}-P_{33}Q_{11}/P_{11})y + (\delta_{33}-P_{33}\delta_{11}/P_{11})$ in this phase the 'remainder' $r_{33}(y)$. Similarly, re-express all other phases in terms of η_{11} and calculate the remainders $r_{13}(y)$, $r_{31}(y)$ and $r_{1133}(y)$. Note that the remainders are, at most, linear expressions of y.

Step 3. Calculate the interval of values for the independent variable y in which all remainders are negative. Concretely, solve the simultaneous inequalities

$$\begin{cases} r_{33}(y) < 0 \\ r_{13}(y) < 0 \\ r_{31}(y) < 0 \\ r_{1133}(y) < 0. \end{cases}$$

The above interval corresponds to a particular subset of the line $\zeta_{11} = 0$ (obviously, this subset can only be of three types: a half-line, a line-segment or the empty set) on which this phase dominates all other phases (which are all negative). In fact, as will be explained in detail in section 3.2, this subset of the line ζ_{11} acts as a separator between the two domains of the xy-plane where either the phase '0' or the phase ζ_{11} dominates all other phases $\zeta_{11}, \zeta_{13}, \zeta_{33}, \zeta_{1133}$ that appear in (25). We thus have the following:

Claim. *The subset of $\zeta = \zeta_1 - \zeta_2 = 0, \zeta_1, \zeta_2 \in \{0, \zeta_{11}, \zeta_{13}, \zeta_{31}, \zeta_{33}, \zeta_{1133}\}$ calculated above, acts as a boundary between the domains of the xy-plane where the two phases ζ_1, ζ_2 (respectively) dominate all other phases.*

Furthermore, in a (small) region of the xy-plane around this subset of $\zeta = 0$, the two-soliton solution can be approximated by a one-soliton solution with phase ζ.

If, for example, we obtained the half-line $\zeta_{11} = 0|_{y>a}$ in step 3, the two-soliton solution on $\zeta_{11} = 0|_{y>a}$ is approximated by a one-soliton solution having the phase ζ_{11}. If, however, a line-segment $\zeta_{11} = 0|_{a<y<b}$ was obtained, we have a similar result on $\zeta_{11} = 0|_{a<y<b}$. In the case only the empty set was obtained, no soliton with phase ζ_{11} will exist at that time.

Step 4. Repeat the above procedure for the remaining phases. In this way one can obtain a set of lines or line segments that will act as boundaries between different domains in the xy-plane where a certain phase among $\{0, \zeta_{11}, \zeta_{13}, \zeta_{31}, \zeta_{33}, \zeta_{1133}\}$ dominates the others. On these boundaries the original two-soliton solution is approximated by a one-soliton solution.

It goes without saying that the above analysis can be readily extended to general N-soliton solutions.

3.2. Examples and the relation between the phase functions and solitons

We give some examples of our method, showing the relation between the phases and the solitons.

As a first example we choose the parameter values

$$\begin{aligned} &(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) = \left(2, \tfrac{3}{2}, 1, \tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{2}, -1, -\tfrac{4}{3}\right) \\ &(\alpha_1, \alpha_3, \beta_1, \beta_3) = (1, 1, 1, 1) \end{aligned} \tag{33}$$

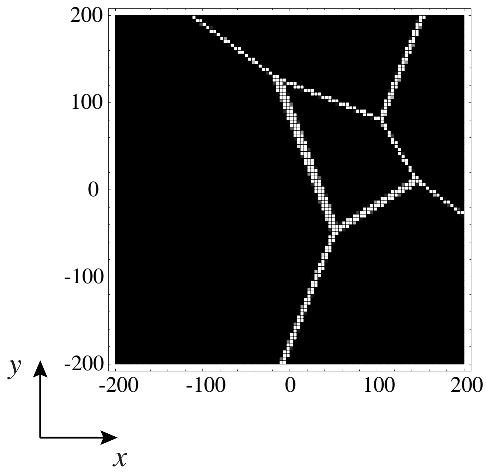

Figure 4. An example of the two-soliton solution with parameters (33) at time $t = -60$ (as a density plot).

and $t = -60, -0.7, 60$. Going through the above scheme at time $t = -60$, we obtain the following result:

$$\begin{aligned}
\zeta_{11}&: y > 129.49\\
\zeta_{33}&: y < -47.62\\
\zeta_{31}&: -47.62 < y < 129.49\\
\zeta_{1133} - \zeta_{11}&: y > 81.29\\
\zeta_{1133} - \zeta_{33}&: y < 11.20\\
\zeta_{1133} - \zeta_{31}&: 11.20 < y < 81.29\\
\zeta_{11} - \zeta_{31}&: 81.29 < y < 129.49\\
\zeta_{31} - \zeta_{33}&: -47.62 < y < 11.20.
\end{aligned}$$

The edges of these intervals can be thought of as 'branch points' in the approximation, i.e. points where transitions between different one-soliton waves occur. We list the coordinates of these branch points:

$$(x, y) = (-14.95, 129.49), (52.52, -47.62), (105.55, 81.29), (147.27, 11.20).$$

It is clear from figure 4 that our analysis correctly predicts the shape of the tetragon one observes: it yields (accurate) phase functions for the one-solitons by which the interaction waves are approximated, as well as the correct locations of the branch points in the tetragon. In other words, we are able to correctly predict what pattern the soliton solution will form in the xy-plane.

The exact relationship between the phase functions appearing in the exponential functions that make up a particular τ and the solitons that are described by it can be understood as follows. Taking the two-soliton τ as an example, one can say that $\log\tau$ is roughly given by the (positive) maximum of the phases, i.e. by $\max(0, \zeta_{11}, \zeta_{33}, \zeta_{13}, \zeta_{31}, \zeta_{1133})$. It is clear, however, that this picture should break down near the boundaries of the domains where each phase dominates. This will be where the actual waves are located. For example, the surface $\log\tau$ plotted in figure 5 is well approximated by the combination of the planes obtained from the maximization $\max(0, \zeta_{11}, \zeta_{33}, \zeta_{13}, \zeta_{31}, \zeta_{1133})$. The solitons and their interactions appear on the aforementioned boundaries, where the surface bends (remember that u is the second

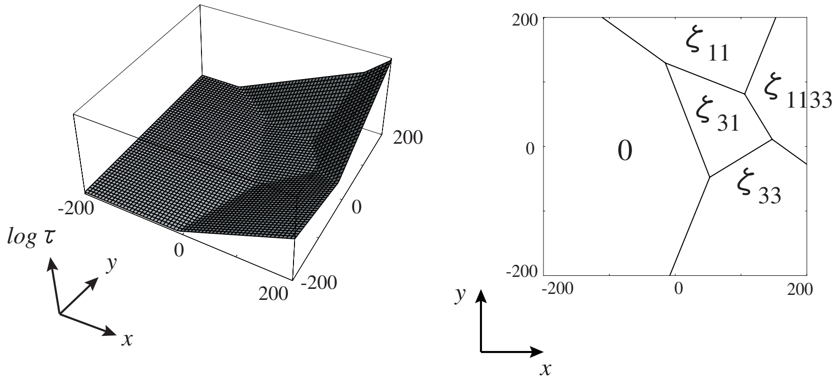

Figure 5. Breakdown of the xy-plane into domains where the respective phase functions dominate; the choice of the parameters and time is that of figure 4.

order derivative of $\log \tau$, i.e. related to the curvature of the surface $\log \tau$). The phase ζ of each soliton is given by the difference of the phase functions ζ_1 and ζ_2 in the domains that have the line $\zeta = 0$ as their common boundary. In the second graph of figure 5 we plotted the domains where each phase dominates and the boundaries $\zeta = \zeta_1 - \zeta_2$ that arise between them. Hence, in order to predict the pattern a particular multi-soliton solution will form in the xy-plane, it is sufficient to locate the boundaries of the domains in the xy-plane where each phase function becomes dominant. This is exactly what the method described in the previous subsection accomplishes. In the case of a two-soliton solution, the xy-plane is divided, at most, into six parts as a two-soliton solution has exactly six phase functions (counting '0' as a phase). However, only five such domains appear in figure 5, leaving one to wonder what has happened to the remaining phase. To answer this question one has to look at what happens to the tetragon at different times.

Applying the same analysis as before at time -0.7, it turns out that the number of line segments found is much greater than before:

$$\begin{aligned}
\zeta_{11}&: 1.5070 < y\\
\zeta_{33}&: y < -0.3644\\
\zeta_{31}&: -0.3644 < y < 1.5070\\
\zeta_{1133} - \zeta_{11}&: 1.4634 < y\\
\zeta_{13} - \zeta_{33}&: -0.1449 < y < 0.4976\\
\zeta_{31} - \zeta_{33}&: -0.3644 < y < 0.4976\\
\zeta_{1133} - \zeta_{33}&: y < -0.1449\\
\zeta_{11} - \zeta_{13}&: 0.7371 < y < 1.4634\\
\zeta_{1133} - \zeta_{13}&: -0.1449 < y < 1.4634\\
\zeta_{11} - \zeta_{31}&: 0.7371 < y < 1.5070\\
\zeta_{13} - \zeta_{31}&: 0.4976 < y < 0.7371.
\end{aligned}$$

The corresponding branch points are

$$\begin{aligned}
(x, y) = {}&(1.096, 1.5070), (1.809, -0.3644), (4.191, 1.4634),\\
&(4.804, -0.1449), (3.198, 0.4976), (3.021, 0.7371).
\end{aligned}$$

In figure 6 we now clearly see that a sixth domain has appeared in the interaction region, a result that is clearly borne out by our analysis but which is much harder to spot visually.

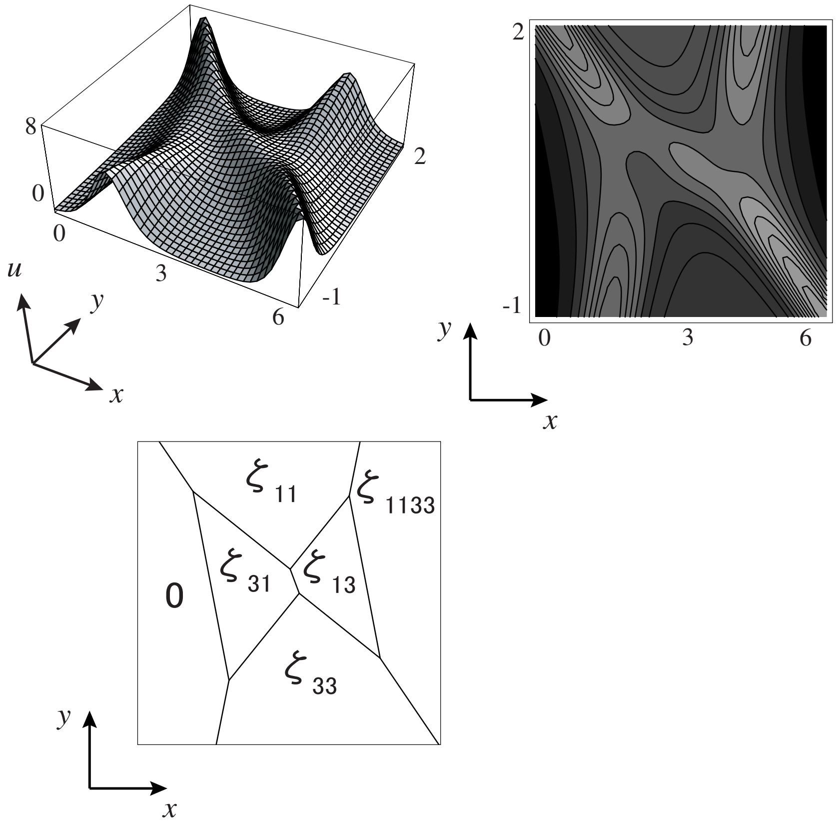

Figure 6. Two-soliton for the parameters of figure 4 (plotted at $t = -0.7$ as a 3D plot and a contour plot) and the breakdown of the xy-plane into dominant phase domains.

As time evolves further, for example at time $t = 60$, we revert to a situation where only five different domains appear in the xy-plane. More precisely, we find the branch points

$$(x, y) = (-144.52, -10.26), (-101.87, -81.91), (-47.72, 49.83), (21.23, -131.15).$$

and a set of dominant domains as in figure 7. Note that in this case the phase ζ_{13} has appeared at the expense of the phase ζ_{31}. Hence, it is clear that the number of phases that appear (or the identity of the phases that appear) may vary over time.

As a second example we take

$$\begin{aligned} &(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) = \left(1, \tfrac{1}{20}, \tfrac{1}{30}, \tfrac{1}{40}, -\tfrac{1}{50}, -\tfrac{1}{43}, -\tfrac{1}{30}, -\tfrac{1}{2}\right) \\ &(\alpha_1, \alpha_3, \beta_1, \beta_3) = (1, 1, 1, 1) \end{aligned} \tag{34}$$

taken at $t = 100$ (see figure 8). We find the following branch points:

$$(x, y) = (-52.05, -42.49), (-46.31, -54.60), (45.60, 148.82), (47.93, -147.98)$$

the dominant phases are given in figure 8. Our analysis shows that this solution also has a web-like structure. At first sight however, one might get the impression from figure 8 that this solution describes a usual two-soliton interaction. This is due to the very small amplitude of the soliton with phase $\zeta_{1133} - \zeta_{13}$ and to the almost negligible differences between the directions of the solitons corresponding to the phases $\zeta_{11} - \zeta_{13}$ and $\zeta_{1133} - \zeta_{11}$, or $\zeta_{13} - \zeta_{33}$ and $\zeta_{1133} - \zeta_{33}$.

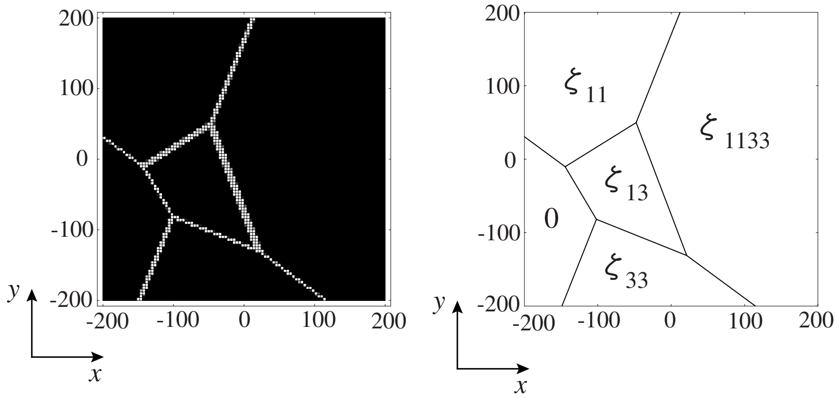

Figure 7. A density plot of the same two-soliton solution as in figure 4, at $t = 60$, together with the distribution of dominant phases in the xy-plane.

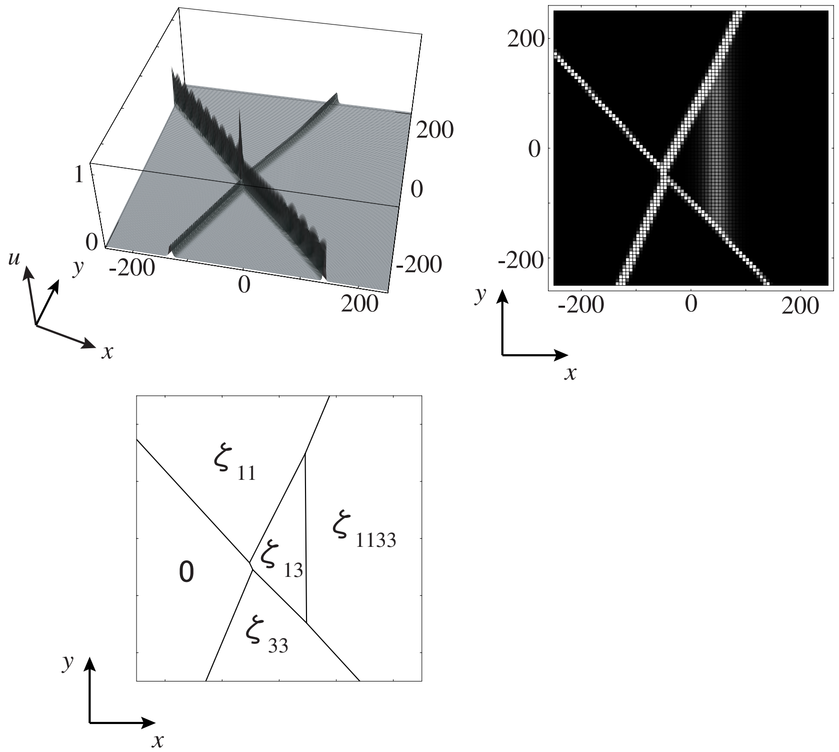

Figure 8. Another example of the two-soliton solution: 3D plot and density plot.

This example shows that visual inspection, due to its inherent resolution-dependence, does not always allow one to appreciate the intricacies of the interaction properties of solitons.

Closing this section we would like to point out that nothing in the present method is specific to the analysis of spider-web solutions. As it stands, the method can be applied with

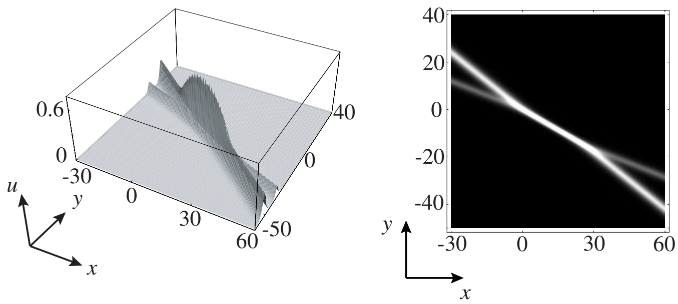

Figure 9. An example of the two-soliton solution of the KP equation: 3D plot and density plot.

equal ease to other sech^2 soliton-type interactions as well. Let us, for example, take the case of the KP equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{35}$$

for which we analyse its two-soliton solution

$$\begin{aligned}\tau &= 1 + \exp\left(\xi_{p_1} - \xi_{q_1} + d_1\right) + \exp\left(\xi_{p_2} - \xi_{q_2} + d_2\right) + \exp\left(\theta_{p_1p_2} - \theta_{q_1q_2} + d_1 + d_2 + \tilde{d}\right)\\ &=: 1 + e^{\hat{\zeta}_1} + e^{\hat{\zeta}_2} + e^{\hat{\zeta}_{12}}\end{aligned} \tag{36}$$

$$e^{\tilde{d}} := \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} \tag{37}$$

where $u = 2(\log\tau)_{xx}$. As an example, we choose the parameters as

$$(p_1, p_2, q_1, q_2) = \left(1.0001, 1, \tfrac{3}{2}, \tfrac{1}{4}\right)$$
$$(d_1, d_2) = (0, 0)$$

and fix time at $t = 0$. Our analysis yields

$$\begin{aligned}\hat{\zeta}_1&: y > 0\\ \hat{\zeta}_2&: y > 0\\ \hat{\zeta}_{12} - \hat{\zeta}_2&: y < -21.35\\ \hat{\zeta}_{12} - \hat{\zeta}_1&: y < -21.35\\ \hat{\zeta}_1 - \hat{\zeta}_2&: -21.35 < y < 0\end{aligned}$$

and the two branch points:

$$(x, y) = (0, 0), (37.36, -21.35)$$

which fits well with the snapshot presented in figure 9.

We also see from this example that, as there are only six phases in the two-soliton solution of the KP equation (the above five plus $\hat{\zeta}$) and as at least eight phases would be needed to realize a spider-web solution, such solutions can never be realized by means of KP two-soliton solutions. We would dare to conjecture that spider-web solutions, such as those put into evidence in figure 3, should actually be regarded as a particular feature of coupled systems and cannot be obtained from standard equations such as the KP equation (see also the discussion in section 4.2).

4. The asymptotic behaviour of the cKP two-soliton solution

In this section, we shall discuss the asymptotic behaviour of the cKP two-soliton solutions on the xy-plane at arbitrary times (fixed), assuming that no resonances occur (i.e. that δ_{ij} and δ_{1133} are finite). We also assume that all remainders (as defined in section 3.1) depend on the variable y.

4.1. Asymptotics

We start by proving the following proposition regarding the asymptotic behaviour of the cKP two-soliton.

Proposition 1. *If a soliton with a phase ζ exists in the asymptotic limit $y \to +\infty$ (or $y \to -\infty$) (i.e. localized on the line $\zeta = 0$) at an arbitrary (but fixed) time, a similar soliton (i.e. with the same phase but shifted by a constant) will exist in the asymptotic limit $\to -\infty$ (or $y \to +\infty$) at that same instant.*

Proof. Without loss of generality, we can assume that the soliton having the phase ζ_{11} exists in the asymptotic limit $y \to -\infty$ on the line $\zeta_{11} = 0$. This assumption is equivalent to the inequalities

$$Q_{33} - P_{33}Q_{11}/P_{11} > 0 \tag{38}$$

$$Q_{13} - P_{13}Q_{11}/P_{11} > 0 \tag{39}$$

$$Q_{31} - P_{31}Q_{11}/P_{11} > 0. \tag{40}$$

From the table of (26) it is clear that we have to show a soliton with $\zeta_{1133} - \zeta_{33}$ exists in the asymptotic limit $y \to +\infty$, localized on the line $\zeta_{1133} - \zeta_{33} = 0$. This is equivalent to showing that the following inequalities hold:

$$-Q_{33} - (-P_{33})Q_{11}/P_{11} < 0 \tag{41}$$

$$Q_{11} - Q_{33} - (P_{11} - P_{33})Q_{11}/P_{11} < 0 \tag{42}$$

$$Q_{13} - Q_{33} - (P_{13} - P_{33})Q_{11}/P_{11} < 0 \tag{43}$$

$$Q_{31} - Q_{33} - (P_{31} - P_{33})Q_{11}/P_{11} < 0. \tag{44}$$

(Please bear in mind that in this analysis the functions τ and $\hat{\tau} = \tau/\mathrm{e}^{\zeta_{33}}$ are gauge equivalent and hence, correspond to the same solution of the cKP equation.) Clearly, (41) is trivial because of (38). Secondly, (42) is also trivial because it is actually the same expression as (41). Finally, the left-hand side of (43) can be changed to $-(Q_{31} - P_{31}Q_{11}/P_{11})$ by eliminating P_{33} and Q_{33} by (31) and (32). This expression is found to be negative because of (40). The remaining inequality (44) is proven in a similar way. □

Next, we study and classify the possible asymptotic behaviour of the two-soliton solution. We again assume that a soliton with a phase ζ_{11} exists in the asymptotic limit $y \to -\infty$ on the line $\zeta_{11} = 0$, i.e. (38)–(40) are satisfied. We shall be interested in knowing which other phases are asymptotically compatible with this assumption. Since $\zeta_{11} + \zeta_{33} = \zeta_{13} + \zeta_{31}$, up to a constant, we can consider either ζ_{13} or ζ_{31} as phases which can be determined at will, making ζ_{33} special. In the following we shall therefore study the (asymptotic) occurrence of ζ_{33} and ζ_{13}.

First, let us consider the case of ζ_{33}. For a soliton having the phase ζ_{33} to exist asymptotically it is required that the expressions

$$Q_{11} - P_{11}Q_{33}/P_{33} \tag{45}$$

$$Q_{13} - P_{13}Q_{33}/P_{33} \tag{46}$$

$$Q_{31} - P_{31}Q_{33}/P_{33} \tag{47}$$

all have the same sign. In order to deal with these inequalities, it turns out that we have to make certain assumptions about the respective signs of the parameters, namely: (i) $P_{11} > 0$, $P_{33} > 0$; (ii) $P_{11} > 0$, $P_{33} < 0$; (iii) $P_{11} < 0$, $P_{33} > 0$; (iv) $P_{11} < 0$, $P_{33} < 0$.

Case (i): The fact that (45) is negative follows from (38): $Q_{11}/P_{11} < Q_{33}/P_{33}$. Hence, (46) and (47) must be negative. Then, if $P_{13} < 0$, we obtain $Q_{13}-P_{13}Q_{33}/P_{33} > Q_{13}-P_{13}Q_{11}/P_{11}$ but this leads to a contradiction: (46) is positive because of (39). Therefore P_{13} has to be positive. Similarly, we find P_{31} must be positive.

Case (ii): We find that (45) is positive from (38). Thus (46) and (47) must be positive as well. Unlike case (i) we cannot find trivial inconsistencies, even if we make some more assumptions about the signs of parameters.

Case (iii): This case is similar to case (ii) and the details are omitted.

Case (iv): We also omit the detailed description of this case, as it is very similar to case (i).

We arrange these results in the following proposition.

Proposition 2. *Provided that the soliton with the phase ζ_{11} exists in the asymptotic limit $y \to -\infty$ (on the line $\zeta_{11} = 0$), the soliton with the phase ζ_{33} will exist:*

(i) in the asymptotic limit $y \to +\infty$ (on the line $\zeta_{33} = 0$) if and only if the parameters satisfy (38)–(40) and

$$P_{ij} > 0 \qquad (i, j = 1, 3) \tag{48}$$

$$Q_{13} - P_{13}Q_{33}/P_{33} < 0 \tag{49}$$

$$Q_{31} - P_{31}Q_{33}/P_{33} < 0 \tag{50}$$

or

$$P_{ij} < 0 \qquad (i, j = 1, 3) \tag{51}$$

$$Q_{13} - P_{13}Q_{33}/P_{33} < 0 \tag{52}$$

$$Q_{31} - P_{31}Q_{33}/P_{33} < 0. \tag{53}$$

(ii) in the asymptotic limit $y \to -\infty$ (on the line $\zeta_{33} = 0$) if and only if the parameters satisfy (38)–(40) and

$$P_{11} > 0,\ P_{33} < 0 \tag{54}$$

$$Q_{13} - P_{13}Q_{33}/P_{33} > 0 \tag{55}$$

$$Q_{31} - P_{31}Q_{33}/P_{33} > 0 \tag{56}$$

or

$$P_{11} < 0,\ P_{33} > 0 \tag{57}$$

$$Q_{13} - P_{13}Q_{33}/P_{33} > 0 \tag{58}$$

$$Q_{31} - P_{31}Q_{33}/P_{33} > 0. \tag{59}$$

Note that we do not know how to choose parameters $\{p_j, q_j\}$ $(j = 1, 2, 3, 4)$ so as to realize the above conditions in any systematic way.

Next, let us consider the appearance of the phase ζ_{13}. In this case we should study the sign of the expressions

$$Q_{11} - P_{11}Q_{13}/P_{13} \tag{60}$$

$$Q_{33} - P_{33}Q_{13}/P_{13} \tag{61}$$

$$Q_{31} - P_{31}Q_{13}/P_{13}. \tag{62}$$

Though we omit the detailed analysis (which is similar to the preceding one), we obtain the following proposition.

Proposition 3. *Provided that the soliton with the phase ζ_{11} exists in the asymptotic limit $y \to -\infty$ (on the line $\zeta_{11} = 0$), the soliton with the phase ζ_{13} exists:*

(i) in the asymptotic limit $y \to +\infty$ (on the line $\zeta_{13} = 0$) if and only if the parameters satisfy (38)–(40) and the conditions obtained from (48)–(50) or (51)–(53) by interchanging subscripts 13 and 33.

(ii) in the asymptotic limit $y \to -\infty$ (on the line $\zeta_{13} = 0$) if and only if the parameters satisfy (38)–(40) and the conditions obtained from (54)–(56) or (57)–(59) by interchanging subscripts 13 and 33.

4.2. Discussion

In this section it will be shown that spider-web solutions appear in the case covered by proposition 2 and that in the case of proposition 3 the two-soliton solution describes what we shall call an essentially three-body collision.

Let us assume the conditions in proposition 2(i) are satisfied. Saying a soliton exists on the line $\zeta_{11} = 0$ in the limit $y \to -\infty$ is equivalent to saying that in this limit the planes ζ_{11} and 0 must have a common boundary (as was demonstrated in section 3.2). Similarly, the planes ζ_{33} and 0 must be adjacent as $y \to +\infty$. Moreover, bearing in mind the identifications (26), from proposition 1 we find that the planes ζ_{33} and ζ_{1133} are adjacent when $y \to +\infty$ and that ζ_{11} and ζ_{1133} are adjacent when $y \to -\infty$. Arranging these four (half-)planes, namely, $0, \zeta_{11}, \zeta_{33}$ and ζ_{1133}, on the xy-plane, it is clear that they fill the entire plane except for a finite domain. It should be clear that only the remaining two phases ζ_{13} and ζ_{31} can appear inside this finite domain. In fact, the expression $\max(0, \zeta_{11}, \zeta_{33}, \zeta_{13}, \zeta_{31}, \zeta_{1133})$ allows us to establish which phases actually arise in this domain. For example, in the first example of section 3.2, ζ_{31} (figure 5) appears on the finite domain at negative time and ζ_{13} at positive time (figure 7). They appear at the same time around $t = 0$ (figure 6). It is exactly the appearance of these phases on this (non-asymptotic) domain that is responsible for the creation of a 'web'-like pattern. Possibly one also has to allow for cases where the 'non-asymptotic phases' never appear or where only one such phase appears all the time.

Next, let us assume the conditions in proposition 3(i) are satisfied. We then have that the planes ζ_{11} and 0 are adjacent for $y \to -\infty$ and ζ_{13} and 0 for $y \to +\infty$. Moreover, from table (26) and proposition 1 we know that the planes ζ_{33} and ζ_{1133} must be adjacent when $y \to +\infty$ and that ζ_{31} and ζ_{1133} are so when $y \to -\infty$. Unlike the discussion in the previous paragraph, we thus find that all six phases (including '0') do appear in the asymptotic limits. In other words, these six phases fill the entire xy-plane and thus no 'web'-like structure is formed. Moreover, one must conclude that, under the above conditions, three solitons exist in each asymptotic limit $y \to \pm\infty$, as can be seen in figure 10. We can therefore make

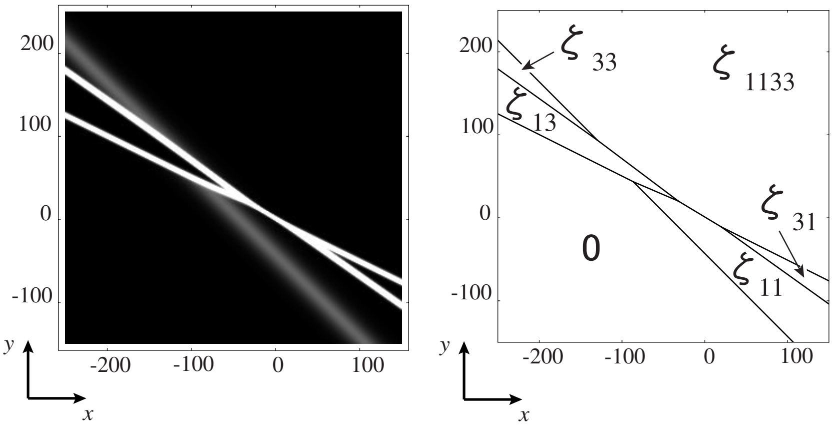

Figure 10. An example of an essentially three-body collision for parameters: $(p_1, p_2, p_3, p_4) = ((-11 - \sqrt{151})/12, (-11 + \sqrt{151})/12, (-29 - \sqrt{4519})/48, (-29 + \sqrt{4519})/48)$), $(q_1, q_2, q_3, q_4) = ((-47 - \sqrt{1999})/48, (-47 + \sqrt{1999})/48, -4/3, -1)$, $(\alpha_1, \alpha_3, \beta_1, \beta_3) = (1, 1, 1, 1)$.

the interesting observation that the two-soliton solution of the cKP equation is rich enough to describe 'three-soliton interactions', for a certain range of parameters (one could argue that the solution should therefore actually be called a 'three-soliton solution'; however, as algebraically speaking the present solution is the simplest one that describes the interaction of two cKP one-soliton solutions, we think it natural to keep on calling it a two-soliton solution).

More importantly, compared to the usual three-soliton solutions, such as those of the KP equation, the present solutions enjoy a remarkable property that hitherto has never been observed for three-soliton solutions. It is a well-known fact that standard N-soliton solutions (such as those for the KP equation) exhibit a remarkable 'two-ness'. For example, even though temporarily (i.e. for a narrow window in time) the spatial (xy-) pattern of a three-soliton interaction for KP solitons might look as if only 'new' intermediate waves appear—unrelated to the ingoing and outgoing solitons—over time 'shifted' (in space) versions of the original solitons will always appear as well. Therefore, for most of the time, the spatial pattern one observes is nothing but a sequence of successive two-soliton interactions, for exactly three solitons. At each 'branch' in the pattern the solitons will pick up phase shifts due to a two-soliton interaction and hence one can conclude that the total phase shift each soliton undergoes is exactly the combination of the phase shifts provoked by the two-soliton interactions it underwent. However, in the present case, it is clear from our analysis that every phase associated with a soliton that extends to infinity in the xy-plane appears exactly twice (as can be seen from figure 10), the intermediate solitons thus always corresponding to different phases. The three solitons therefore always undergo a complicated interaction (and this at all times as no solitons disappear) in which three intermediate waves are created, all of them distinct from the solitons that survive asymptotically. Hence, each soliton really undergoes only a single phase shift, in a single interaction. Furthermore, as there are no two-soliton interactions happening (not distinguishable as such in any case) the total phase shift incurred by each soliton cannot (obviously) be broken down into a sum of elementary ones. We therefore call this collision of three solitons an *essentially three-body collision*.

According to the discussion above, the two-soliton solution of the cKP equation exhibits two different types of patterns, depending on the choice of parameters. It can be seen that patterns can move from one such type to the other. This is because there exist choices of

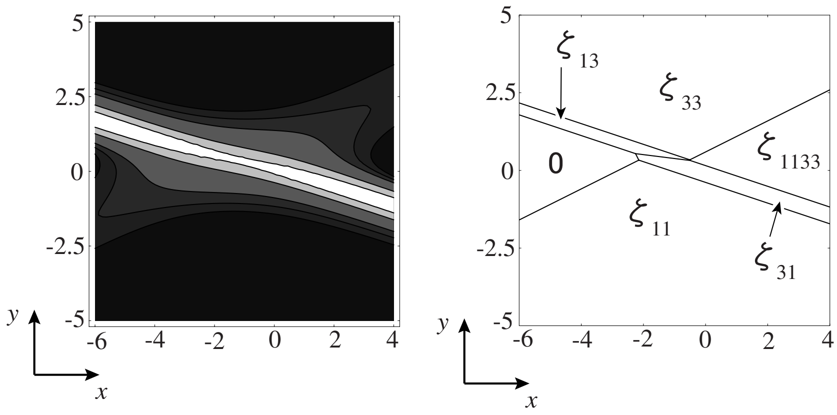

Figure 11. The two-soliton solution for the parameter values (65), contour plot and dominant phases.

the parameters that satisfy both sets of conditions. Namely, the following condition on the parameters

$$\begin{cases} P_{ij} > 0 \\ Q_{11}/P_{11} < Q_{31}/P_{31} < Q_{33}/P_{33} = Q_{13}/P_{13} \end{cases} \tag{63}$$

acts as an intermediate constraint between the conditions in propositions 2(i) and 3(i). Let us choose some parameters which realize (63) for a set of given values of P_{jk}, Q_{jk}. For example, satisfying (63) with P_{jk}, Q_{jk} as

$$\begin{array}{llll} P_{11} = \frac{1}{2} & P_{33} = 2 & P_{13} = \frac{3}{2} & P_{31} = 1 \\ Q_{11} = -1 & Q_{33} = 6 & Q_{13} = \frac{9}{2} & Q_{31} = \frac{1}{2} \end{array} \tag{64}$$

one can obtain parameter values for p_j, q_j $(j = 1, 2, 3, 4)$ by solving (28), (29) with (64). It turns out that two of the eight parameters are arbitrary (this arbitrariness can actually be used to avoid divergences in the field u). This is shown in the example:

$$\begin{cases} (p_1, p_2, p_3, p_4) = \left(\frac{5}{2}, \frac{3}{4}, \frac{15+\sqrt{41}}{8}, \frac{15-\sqrt{41}}{8}\right) \\ (q_1, q_2, q_3, q_4) = \left(\frac{11+\sqrt{129}}{8}, \frac{11-\sqrt{129}}{8}, \frac{1}{4}, \frac{3}{2}\right). \end{cases} \tag{65}$$

We analyse the behaviour of the two-soliton solution that arises from the above parameter values (together with $(\alpha_1, \alpha_3, \beta_1, \beta_3) = (1, 1, 1, 1)$) at $t = 0$, resulting in the branch points:

$$(x, y) = (-2.1495, 0.3307), (-2.2539, 0.5396), (-0.5061, 0.3414), (-0.4953, 0.3198)$$

and a distribution of dominant phases as in figure 11. Two of the three resulting solitons turn out to be parallel in the xy-plane and are nearly indistinguishable on a graphical plot (see the left-hand plot in figure 11).

At $t = -60$ however, the result of our analysis is as follows:

$$\begin{aligned} \zeta_{11}&: y < 240.435 \\ \zeta_{13}&: y > 240.435 \\ \zeta_{1133} - \zeta_{11}&: y < 238.658 \\ \zeta_{13} - \zeta_{33}&: y > 243.341 \\ \zeta_{1133} - \zeta_{33}&: y > 243.341 \\ \zeta_{11} - \zeta_{13}&: 238.658 < y < 240.435 \\ \zeta_{1133} - \zeta_{13}&: 238.658 < y < 243.341 \end{aligned}$$

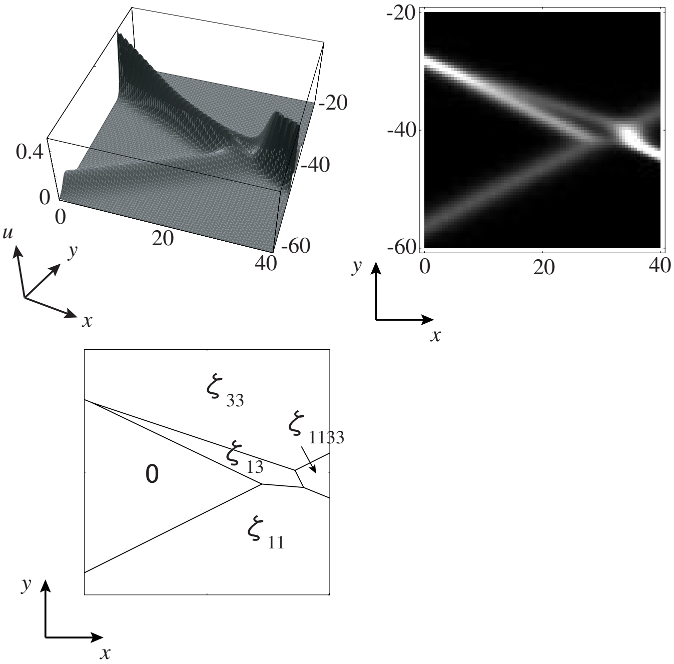

Figure 12. The two-soliton solution for the parameter values (66): 3D plot, density plot and phase diagram.

whereas at $t = 60$ we find:

$$\begin{aligned}
\zeta_{11}&: y < -242.669\\
\zeta_{33}&: y > -237.845\\
\zeta_{31}&: -242.669 < y < -237.845\\
\zeta_{31} - \zeta_{33}&: -239.669 < y < -237.845\\
\zeta_{1133} - \zeta_{33}&: y > -239.669\\
\zeta_{11} - \zeta_{31}&: y < -242.669\\
\zeta_{1133} - \zeta_{31}&: y < -239.669
\end{aligned}$$

We thus see that the phase function ζ_{31} does not appear at $t = -60$ but does at $t = 60$ and vice versa for ζ_{13}. Although in an intermediate time range we clearly have a three-body type solution—two solitons in which are parallel—allowing for 'infinite boxes' one could argue that this solution is also of spider-web type. Hence our belief that the interactions that make up spider-web solutions are not of the usual 'two-soliton' type, but are essentially multi-body type interactions. This would explain why such structures have never been observed in soliton systems like the KP equation.

Finally, note that changing the parameters in the above solution slightly, for example, as

$$\begin{cases}(p_1, p_2, p_3, p_4) = \left(\frac{5}{2}, \frac{3}{4}, \frac{15+\sqrt{41}}{8}, \frac{15-\sqrt{41}}{8}\right)\\ (q_1, q_2, q_3, q_4) = \left(\frac{11+\sqrt{129}}{8}, \frac{11-\sqrt{129}}{8}, \frac{1}{4}, \frac{3}{2} + 0.8\right)\end{cases} \tag{66}$$

the solution becomes purely web-like; the analysis at $t = 10$ (see also figure 12) yielding the branch points

$$(x, y) = (28.938, -41.938), (1.2887, -28.682), (35.747, -42.492), (34.353, -39.703).$$

and a phase diagram as in figure 12.

5. Summary

In this paper we introduced a novel method for analysing soliton interactions on finite domains. The method basically amounts to an approximation of the solitons and their interaction waves in terms of individual one-solitons and yields accurate estimates for the spatial extent of each wave. As an analytic method, it also allowed us to predict in which regions of space individual solitons existed even in the cases where numerical or graphical inspections failed to provide such information.

The method was applied to the so-called spider-web solutions for the cKP equation, allowing us to identify the exact phases involved in the interactions that make up the threads of a web. It also allowed us to discover the existence of what we called essentially three-body interactions for the cKP solitons, i.e. multi-soliton interactions that cannot be broken down into sequences of two-soliton interactions. It should be stressed that both types of solutions are quite different from what is found in the literature on standard KP-like soliton interactions, where N-solitons (see e.g. [10–12] for relevant details on KP-type soliton interactions) might give rise to complicated interaction patterns that—their complexity notwithstanding—are always the result of a succession of (possibly resonant) two-soliton interactions. It should be noted, however, that structures very much similar to the web-type solutions we discuss have been shown to exist in interactions of dissipatons for a particular NLS equation [13]. As these are solutions for a $(1 + 1)$-dimensional system, the web-like structures appear in the space–time domain. However, the existence of such solutions is not very surprising, as both the NLS equation and the present cKP system find their origins in the two-component KP hierarchy. This suggests that web-like structures might be a common feature for the equations in that hierarchy or their dimensional reductions. It therefore seems worthwhile to apply the analytic method we proposed here for other types of solitons than the KP-type solitons (i.e. non-sech^2), especially in the setting of the two-component KP hierarchy and many physically interesting equations in it.

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